Linearity rules of integration

Introduction

To enable us to find integrals of a wider range of functions than those normally given in a Table of Integrals we can make use of two rules known as **linearity rules**.

1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

$$\int k\, f(x)\, \mathrm{d}x = k \int f(x) \mathrm{d}x$$

This is only possible when k is a constant, and it multiplies some function of x.

Example
Find $\int 11x^2\, \mathrm{d}x$.

Solution
We are integrating a multiple of x^2. The constant factor,11, can be moved outside the integral sign.

$$\int 11x^2\, \mathrm{d}x = 11 \int x^2\, \mathrm{d}x = 11\left(\frac{x^3}{3} + c\right) = \frac{11x^3}{3} + 11c$$

where c is the constant of integration. Because $11c$ is a constant we would normally write the answer in the form $\frac{11x^3}{3} + K$ where K is another constant.

Example
Find $\int -5\cos x\, \mathrm{d}x$.

Solution
We are integrating a multiple of $\cos x$. The constant factor, -5, can be moved outside the integral sign.

$$\int -5\cos x\, \mathrm{d}x = -5 \int \cos x\, \mathrm{d}x = -5\,(\sin x + c) = -5\sin x + K$$

where K is a constant.

2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

$$\int f(x) + g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

$$\int f(x) - g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

Example
Find $\int (x^3 + \sin x)\mathrm{d}x$.

Solution

$$\int (x^3 + \sin x)\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{x^4}{4} - \cos x + c$$

Note that only a single constant of integration is needed.

Example
Find $\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x$.

Solution

$$\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x = \int \mathrm{e}^{3x}\,\mathrm{d}x \quad - \quad \int x^7\,\mathrm{d}x = \frac{\mathrm{e}^{3x}}{3} - \frac{x^8}{8} + c$$

Exercises

1. a) Find $\int 8x^5 + 3x^2\,\mathrm{d}x$, b) $\int \frac{2}{3}x\,\mathrm{d}x$.
2. Find $\int 3\cos x + 7x^3\,\mathrm{d}x$.
3. Find $\int 7x^{-2}\mathrm{d}x$.
4. Find $\int \frac{5}{x}\mathrm{d}x$.
5. Find $\int \frac{x+\cos 2x}{3}\mathrm{d}x$.
6. Find $\int 5\mathrm{e}^{4x}\mathrm{d}x$.
7. Find $\int \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\mathrm{d}x$.

Answers

1. a) $\frac{4x^6}{3} + x^3 + c$, b) $\frac{1}{3}x^2 + c$, 2. $3\sin x + \frac{7x^4}{4} + c$, 3. $-\frac{7}{x} + c$ 4. $5\log_{\mathrm{e}}|x| + c$,
5. $\frac{x^2}{6} + \frac{\sin 2x}{6} + c$, 6. $\frac{5\mathrm{e}^{4x}}{4} + c$, 7. $\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} + c$.